DYNAMIC MODELS OF THE HIPP PENDULUM REGULATOR

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L. A. KOMRAZ
(Gor'kii)
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The dynamics of an electromechanical clock with the Hipp movement is considered. Two models are investigated: a dry-friction model and a viscous-friction model. In the case of the model with dry friction we isolate the domains of the parameter space which correspond to simple stable periodic motions, the domains corresponding to complex stable periodic motions, and the domain of parameter values in which the system experiences nonperiodic oscillatory motion. In the case of the viscous-friction model we isolate the domains of the parameter space corresponding to a single stable periodic motion, the domains corresponding to two simple stable periodic motions, the domains corresponding to complex stable periodic motions, and the domains corresponding to two stable periodic motions, one of which is simple and the other complex.

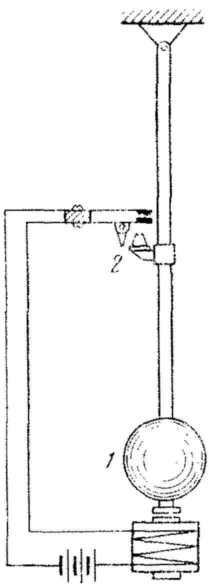

Fig. 1

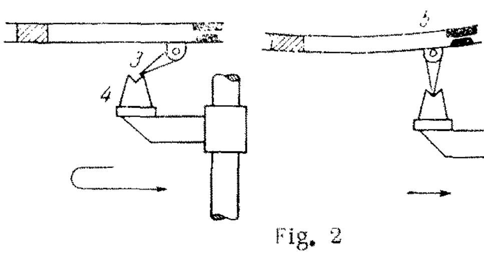

Fig. 2

1. Description of the mechanism. Equations of motion. Conversion to an instantaneous-pulse model. A schematic diagram of the Hipp regulator in an electromechanical clock is shown in Fig. 1.

Pendulum *1* experiences free damped oscillations until contact device *2* closes the electrical circuit and the pendulum receives a pushing pulse which increases its swing.

Figure 2 shows two positions of the contact device. The first position corresponds to the instant when the velocity of the pendulum changes sign, and when the amplitude has decreased to such an extent that tongue *3* mounted on the contact leaf spring is caught between the teeth of catch *4* connected to the pendulum rod. As the pendulum swings back, tongue *3* bends contact spring *5* and closes the electrical circuit (second position in Fig. 2).

Neglecting the damping of oscillator motion during the pulse and making the usual simplifying assumptions concerning the character of pulse transfer [1, 2], we obtain the following equations of motion:

$$mx^{\cdot\cdot} + H_2 x^{\cdot} + kx = -Q\frac{x^{\cdot}}{|x^{\cdot}|}, \qquad y_1 = 0 \tag{1.1}$$

between pulses, and

$$mx^{\cdot\cdot} + kx = My_1^2, \ Ly_1^{\cdot} + Ry_1 = E \tag{1.2}$$

during a pulse.

Here m is the reduced mass; H_2, Q are the coefficients of viscous and dry friction, respectively; k is the restoring force coefficient; My_1^2 is the magnitude of the force; L is the self-inductance of the coil; R is the circuit resistance; E is the electromotive force; x is the coordinate of the pendulum measured from the position of static equilibrium; y_1 is the current strength.

Let $A_1 > \ldots > A_N > \ldots$ be the series of successive "left-hand"amplitudes with free damped motion of the pendulum between two successive pulses. The pulse in conveyed to the pendulum in the zone

$$-2b \leqslant x < 0, \quad x^{\cdot} > 0$$

if the amplitude A_N falls in the interval

$$|m_1| < A_N \leqslant |M_1|$$

where $|M_1|$ is the maximum amplitude at which the contact device is actuated, and

$$m_1 = M_1 \exp\left(-\frac{\pi H_2}{m\omega}\right) + \frac{Q}{k}\left[\exp\left(-\frac{\pi H_2}{2m\omega}\right) + 1\right]^2 \tag{1.3}$$

$$\omega = \frac{1}{2m}(4mk - H_2^2)^{1/2} \qquad (H_2^2 - 4mk < 0)$$

Introducing new variables and the parameters

$$t = \left(\frac{k}{m}\right)^{1/2} tcm, \qquad y = \left(\frac{M}{k}\right)^{1/2} y_1, \qquad A = \frac{R}{L}\left(\frac{m}{k}\right)^{1/2} \tag{1.4}$$

$$B = \frac{E}{R}\left(\frac{M}{k}\right)^{1/2}, \qquad H_1 = \frac{H_2}{2\sqrt{mk}}, \qquad r = \frac{Q}{k}$$

we reduce Eqs. (1.1) and (1.2), respectively, to the form

$$x^{\cdot\cdot} + 2H_1 x^{\cdot} + x = -r\frac{x^{\cdot}}{|x^{\cdot}|}, \qquad y = 0 \tag{1.5}$$

$$x^{\cdot\cdot} + x = y^2, \ y^{\cdot} + Ay = AB \tag{1.6}$$

The changeover from (1.6) to (1.5) occurs when $x = 0$, $x^{\cdot} > 0$;the changeover from (1.5) to (1.6) occurs when $x = -2b$ on the segment s. The segment s is the image of the segment $[M_1, m)$ of the x-axis on the half-line $x = -2b$, $y = 0$, $x^{\cdot} > 0$ along the segments of the trajectories of Eq. (1.5) with the smallest changeover time.

Let us convert to the instantaneous-pulse model by taking the limit as $b \to 0$ in Eqs. (1.6) and setting

$$\int_0^{\tau} y^2 \, d\tau = B^2 \psi(\tau) \equiv I_0 = \text{const}, \ \psi(\tau) = \tau + \frac{2}{A}(e^{-A\tau} - 1) - \frac{1}{2A}(e^{-2A\tau} - 1) \tag{1.7}$$

Here τ is the pulse duration.

To this end we must find analytic expressions for the point transformation of the segment s into the segment σ of the half-line $x = 0$, $x^{\cdot} > 0$, $y = 0$.

Let $u \in s$ and $v \in \sigma$. We then obtain the following expressions for the point transformation:

$$u = \frac{1}{\sin\tau}[2b\cos\tau - B^2\varphi(\tau)], \qquad v = \frac{1}{\sin\tau}[2b + B^2\xi(\tau)] \tag{1.8}$$

Here

$$\varphi(\tau) = 1 - \cos\tau - 2F(A, \tau) + F(2A, \tau), \quad F(A, \tau) = \frac{e^{-A\tau} - \cos\tau + A\sin\tau}{1 + A^2}$$

$$\zeta(\tau) = 1 - \cos\tau - 2\Phi(A, \tau) + \Phi(2A, \tau), \quad \Phi(A, \tau) = \frac{1 - e^{-A\tau}(\cos\tau + A\sin\tau)}{1 + A^2}$$

$$0 < \tau \leqslant \tau_0 < {}^1/_2\,\pi$$

From the first equation of (1.8) we infer that for any $u = \text{const} > 0$ there exists (see Appendix 1) a function $\tau = \tau(b)$ such that $\lim\tau(b) = 0 \quad (b \to 0)$.

Making use of (1.7), we can rewrite (1.8) as

$$u = \frac{1}{\sin\tau}\left[2b\cos\tau - I_0\frac{\varphi(\tau)}{\psi(\tau)}\right], \qquad v = \frac{1}{\sin\tau}\left[2b + I_0\frac{\xi(\tau)}{\psi(\tau)}\right] \tag{1.9}$$

Taking the limit as $b \to 0$ in expressions (1.9) (see Appendix 2),

$$u = {}^1/_4\,[(h / I_0)^{1/3} - I_0], \qquad v = {}^1/_4\,[(h / I_0)^{1/3} + 3I_0]$$

and eliminating I_0, we finally obtain the relationship between the prepulse and postpulse velocities of the oscillator acted on by an instantaneous pulse,

$$(v - u)(v + 3u)^3 = h \tag{1.10}$$

Let us consider curve (1.10) in the plane uv. We have

$$\frac{dv}{du} = \frac{3u - 2v}{v}, \qquad \frac{d^2v}{du^2} = \frac{3h}{v^3(v - 3u)^2} > 0 \tag{1.11}$$

For $u = 0$ we have $dv / du = -2$; curve (1.10) has a minimum for $v = {}^3/_2\,u$. As $u \to \infty$ it approaches the asymptote $v = u$ from above (Fig. 3).

Noting that

$$\frac{dv}{dh} = \frac{1}{4v(v + 3u)}, \qquad \frac{du}{dh} = \frac{1}{4(2v - 3u)(v + 3u)}\begin{cases} > 0 & \text{for } v > {}^3/_2\,u \\ < 0 & \text{for } v < {}^3/_2\,u \end{cases} \tag{1.12}$$

we conclude that as the parameter h increases (decreases) curve (1.10) rises (falls).

2. The dry-friction model. 2.1. Reduction of the problem to a point transformation. Let us consider a model with dry friction (*) (setting $H_1 = 0$ in Eq. (1.5)) and with the segment s contiguous with the axis $x' = 0$. In this case (1.3) and (1.4) imply

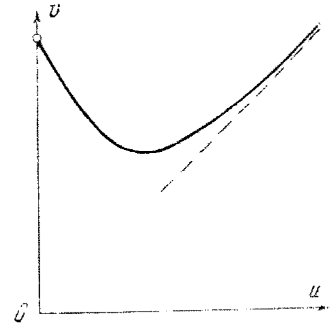

Fig. 3

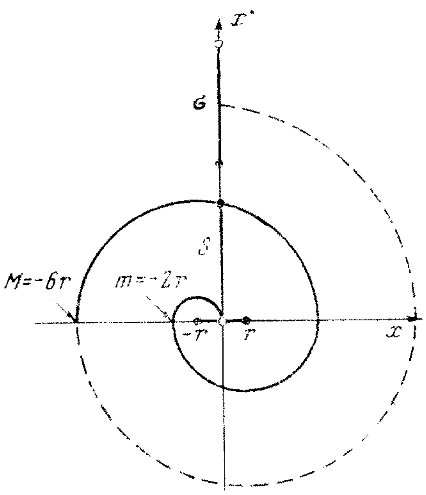

Fig. 4

*) See next page.

that $M_1 = -6r$, $m_1 = -2r$ (Fig. 4).

The points of the segment s $(x = 0,\ o < x^{\cdot} \leqslant 2r\sqrt{6})$ correspond to states in which the oscillator receives a pulse. On reaching the point with the coordinates $x = 0$, $x^{\cdot} = u$ on the segment s, the representing point instantaneously jumps in accordance with Eq. (1.10) to the point $x = 0$, $x^{\cdot} = v$. The segment σ of the half-line $x = 0$, $x^{\cdot} > 0$ is the mapping of s over Eq. (1.10). The phase space of the system is filled with the trajectories of Eq. (1.5) (in the given case by spirals consisting of segments of half-circles) originating at points of the segment σ and ending at points of the segment s. The segment s lies between neighboring coils of the spiral. Every trajectory having a point of σ as its origin intersects the segment s if the segments σ and s do not have points in common (the points belonging to the intersection of σ and s lie on the rest segment). The point transformation T into itself of the segment σ completely defines the structure of the decomposition of the phase space into trajectories.

The transformation of the segment σ into the segment s $(v_1 \in \sigma,\ u \in s)$ is of the following form:

$$(v_1^2 + r^2)^{1/2} - (u^2 + r^2)^{1/2} = 4rN \tag{2.1}$$

Here N is determined unambiguously for each v_1.

2.2. Investigation of the transformation of the segment σ into the segment s. Let us consider curve (2.1) $(N = 1, 2, \ldots)$ in the plane u, v_1. Let D_N $(0,\ v_1(0, N))$ be the points of intersection of the various branches of curve (2.1) with the straight line $u = 0$ $(v_1(0, N) = 2r\,[2N(2N + 1)]^{1/2})$. The points P_N $(u_1,\ v_1(0, N + 1))$ are then the points of intersection of the corresponding branches of curve (2.1) with the straight line $u = u_1$ $(u_1 = 2r\sqrt{6})$. We denote the half-intervals $v_1(0,N) < v_1 < v_1(0,N + 1)$ on the axis v_1 by s_N.

Equation (2.1) defines the quantity u as a single-valued function of v_1 for all $v_1 > v_1(0, 1)$ and u belonging to s $(0 < u \leqslant u_1)$; this function is discontinuous at the points $v_1 = v_1(0, N)$.

From expression (2.1) we obtain

$$\frac{dv_1}{du} = \frac{u}{v_1}\left(\frac{v_1^2 + r^2}{u^2 + r^2}\right)^{1/2} \geqslant 0, \qquad \frac{d^2v_1}{du^2} = \frac{r^2}{u(u^2 + r^2)}\left(\frac{dv_1}{du} - \frac{u^3}{v_1^3}\right) > 0 \tag{2.2}$$

2.3. Investigation of the stability of the fixed points of the transformation T. Now let us consider curves (1.10) and (2.1) (the Lameray diagram) on the plane uv. For the parameter values which satisfy the condition

$$r = {}^1/_{12}\sqrt{6}\,h^{1/4} \tag{2.3}$$

curve (1.10) and the branch $N = 1$ of curve (2.1) cut off equal segements $v = 2r\sqrt{6}$ (i.e. they pass through the point D_1) of the axis v of the Lameray diagram. For $r > {}^1/_{12}\sqrt{6}h^{1/4}$ curves (1.10) and (2.1) do not have points of intersection (here $\sigma \subset s$ and the clock does not work).

Let us take the parameter values which satisfy condition (2.3). At the point D_1 we have $dv / dv_1 = -\infty$. We hold the parameter r fixed and increase h until curves (1.10)

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*) See [3] for a discussion of a clock with the Hipp movement in this idealization with allowance for dry friction only. Postulating the realizability of the simplest type of motion for all values of the system parameters, the author of [3] confines himself to the determination of the period of this motion.

and (2.1) intersect at the point P_1 (such a value of h exists by virtue of (1.12)). At this point we have

$$dv / dv_1 = {}^5/_9\,(3 - \sqrt{{}^{40}/_3}) > -1$$

It follows from the above that there are values of the parameters h and r such that

$$dv / dv_1 = -1$$

at the point of intersection of curves (1.10) and (2.1).

A similar analysis can be carried out for the branch $N = 2$ of curve (2.1). Thus, for the branches $N = 1$ and $N = 2$ there exists a bifurcation of a fixed point of the transformation T for which the root of the characteristic polynomial passes through the value $\lambda = -1$.

The equation of the corresponding bifurcation curve is of the form

$$(v - u)\,(v + 3u)^3 = h, \quad (v^2 + r^2)^{1/2} - (u^2 + r^2)^{1/2} = 4rN \tag{2.4}$$

$$\frac{u}{v}\left(\frac{v^2 + r^2}{u^2 + r^2}\right)^{1/2} = \frac{2v - 3u}{v}$$

From the third expression of (2.4) we obtain

$$r^2 = \frac{3}{4}\,\frac{3u - v}{v - 2u}\,u^2 \tag{2.5}$$

This implies that

$$2 < z < 3 \qquad (z \equiv v / u) \tag{2.6}$$

From (2.4) we readily obtain an equation for determining z .

$$(z + 1)\,(z + 2)^2 - 12N^2\,(3 - z)^2 = 0 \tag{2.7}$$

This equation has a unique positive root. Making use of (2.5), we obtain the following expression for the bifurcation curve:

$$r = \left[\frac{9\,(3 - z)^2\,h}{16\,(z - 2)^2\,(3 + z)^3\,(z - 1)}\right]^{1/4} \equiv r_1\,(z,\ h) \tag{2.8}$$

Here z is the root of Eq. (2.7).

For the coordinates of the point P_3 (the coordinates of the right-hand end of the branch $N = 3$ of curve (2.1)) is $z = 2\sqrt{3} > 3$. For the points of intersection of curve (1.10) with the branches $N = 3, 4, \ldots$ we have $dv / dv_1 < -1$, so that the corresponding fixed points of the transformation T are unstable.

For parameter values which satisfy the condition

$$r = \frac{h^{1/4}}{\{2^6\,[\sqrt{(N + 1)\,(2N + 3)} - \sqrt{3}]\,[3\sqrt{3} + \sqrt{(N + 1)(2N + 3)}]^3\}^{1/4}} \equiv r_2\,(N,\ h) \tag{2.9}$$

curve (1.10) passes through the point P_N. If the parameters chosen satisfy the condition

$$r_1\,(z,\ h) < r < r_2\,(N,\ h) \tag{2.10}$$

(here z is the root of Eq. (2.7)) for $N = 1$ and $N = 2$, then the fixed points of the transformation T are stable.

2.4. Instability of the fixed points of the transformation T^n for $N > 3$. When the parameters chosen satisfy the inequality

$$h^{1/4} \geqslant 6r\,[4\,(2 - \sqrt{3}/3)\,(2 + \sqrt{3})^3]^{1/4} \tag{2.11}$$

for $u \in s$, curve (1.10) lies above the point P_3.

The following statement is valid: for parameters satisfying condition (2.11) the transformation T^n $(n = 1, 2, \ldots)$ can have only unstable fixed points (only unstable limit cycles can exist in the phase space of the dynamic system under consideration).

In fact, for any $u \in s$ we have the inequality

$$\left|\frac{dv_1}{du}\right|_{N=3} > \left|\frac{dv_1}{du}\right|_{N>3}$$

(the slopes of the branches of curve (2.1) decrease with increasing N for each u); fulfillment of (2.11) implies that the following chain of inequalities holds:

$$\min\left|\frac{dv}{du}\right| = \left|\frac{dv}{du}\right|_{P_3} > \left|\frac{dv_1}{du}\right|_{P_3} = \max\left|\frac{dv_1}{du}\right|_{N=3} = \max\left|\frac{dv_1}{du}\right|_{N>3} \qquad (2.12)$$

Let the transformation T^n have a fixed point $v^{(1)} = v^{(n+1)} = v^*$, i.e.

$$v^* = T^n v^*$$

We now have the following expression for the root λ of the characteristic polynomial of the transformation T^n:

$$|\lambda| = \prod_{i=1}^{n}\left|\frac{dv}{du}\right|_{v^{(i)}}\left|\frac{du}{dv_1}\right|_{v^{(i+1)}}$$

But by virtue of (2.12) we have

$$\left|\frac{dv}{du}\right|_{v^{(i)}}\left|\frac{du}{dv_1}\right|_{v^{(i+1)}} > 1$$

Hence, $|\lambda| > 1$ and the statement has been proved.

2.5. The number of fixed points of the transformation T. Curve (1.10) can have only one point of intersection with each branch of curve (2.1). In fact, at the point of intersection ($v = v_1$) we have

$$\frac{dv}{du} = \frac{3u - 2v}{v} < \frac{u}{v} < \frac{u}{v}\left(\frac{v^2 + r^2}{u^2 + r^2}\right)^{1/2} = \frac{dv_1}{du}$$

Let us investigate the number of points of intersection of curve (1.10) with the various branches of curve (2.1) for $u \in s$. When the chosen parameters satisfy condition (2.9), curve (1.10) intersects the Nth branch of curve (2.1) at the point P_N at which (see Appendix 3) $dv / du < 0$. Hence, by virtue of (1.12) among the parameters satisfying the condition

$$r > r_2(N, h)$$

there are parameters such that curve (1.10) intersects not fewer than two (N and $N + 1$) branches of curve (2.1)).

For parameter values satisfying the condition

$$r = \frac{h^{1/4}}{2\,[2\,(N + p + 1)\,(2N + 2p + 3)]^{1/2}} \qquad (2.13)$$

curve (1.10) passes through the point D_{N+p+1} for $p = j$ ($j = 0, 1, \ldots$)

The point of intersection of curve (1.10) with the axis v lies in the interval s_{N+j+1} as p varies in the range $j < p \leqslant j + 1$.

Let us stipulate that curve (1.10) passes through the point P_N and determine the largest number $N + j + 1$ of the interval s_{N+j+1} into which it falls.

Making use of (2.9) and (2.13), we obtain the following equation for determining p:

$$(N + p + 1)^2\,(2N + 2p + 3)^2 = [\sqrt{(N + 1)(2N + 3)} - \sqrt{3}][3\sqrt{3} + \sqrt{(N + 1)(2N + 3)}]^3 \qquad (2.14)$$

The number l of required points of intersection of curves (1.10) and (2.1) can be expressed in terms of p

$$l = [p] + 2$$

Here $[p]$ denotes the whole part of p.

We can show (see Appendix 4) that $1 < p < 3$. Thus, when the chosen parameters satisfy condition (2.11), the transformation T can have two, three or four unstable fixed points.

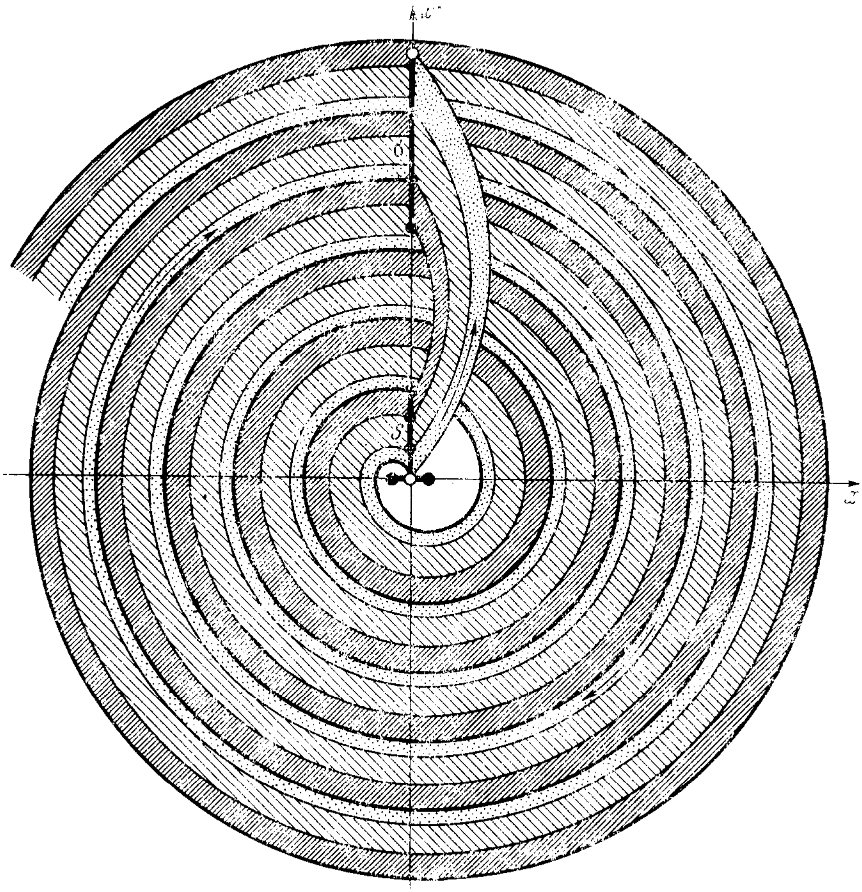

Fig. 5

Figure 5 shows the phase space of the system in the case where curve (1.10) intersects two branches of curve (2.1). The segment s is divided into three parts: the points of the lower part return to s after the jump onto the segment σ (according to (1.10)) and five turns in the plane $xx^{\cdot}$ (according to (2.1)); the points of the middle portion return after four turns, and the points of the upper portion after three turns.

2.6. Decomposition of the parameter space and the character of the fixed points of the transformation T^n. Equation (2.3) defines in the parameter space $h^{1/4}, r$ a straight line which isolates the domain [0] when the clock is not working (Fig. 6).

The requirement that the minimum of curve (1.10) be equal to the minimum of the branch $N = 1$ of curve (2.1) yields the equation

$$r = h^{1/4} / \sqrt{72} \tag{2.15}$$

(the straight line $\{\alpha_1\}$). Above the straight line $\{\alpha_1\}$ in the parameter space is a domain for which the segments σ and s do not have generic points, and such that for any initial conditions the representing point falls into that part of the phase space which is filled with the trajectories of Eq. (1.5) which begin at the segment σ and end at the segment s. In its subsequent motion the representing point cannot leave the indicated portion of the phase plane (the clock is working).

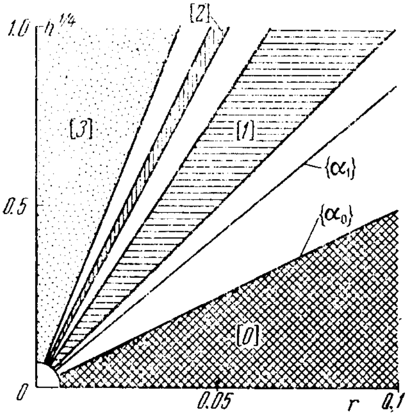

Fig. 6

Inequalities (2.10) (which can be fulfilled for $N = 1$ or $N = 2$ only) isolate from among the parameters lying above the straight line $\{\alpha_1\}$ the domains of existence of stable fixed points of the transformation T corresponding to the simplest stable periodic motion in which a pulse is transferred after each N oscillations of the oscillator (in Fig. 6 domain [1] corresponds to $N = 1$ and domain [2] to $N = 2$).

Condition (2.11) isolates domain [3] which corresponds to nonperiodic oscillatory motion of the system (the representing point does not leave the indicated strip of the phase space in which a stable limit cycle of any multiplicity cannot exist).

Let us consider the domain of the parameter space which is defined by the inequality

$$r > r_1 (z, h^{1/4})$$

Here z is the root of Eq. (2.7) for $N=1$. In this domain a fixed point of the transformation T can be unstable only. For any r there exist values of the parameter h taken from this domain for which there exists a pair of stable fixed points of the transformation T^2, and also values of this parameter for which there exists a pair of unstable fixed points of the transformation T^2 (see Appendix 5). For this reason we expect that variation of the parameter h will be accompanied by bifurcations of a fixed point of the transformation T^2 such that the root of the characteristic polynomial (for this transformation) passes through the value $\lambda = -1$ and the four fixed points of the transformation T^4 either appear or vanish.

We investigated the character of the bifurcations of the fixed points of the transformation T^n with the aid of a BESM-3M computer.

In order to determine the character of the bifurcation of the fixed point of the transformation T for which the root of the characteristic polynomial passes through the value $\lambda = -1$ (passage in the parameter space through the lower boundary of either domain [1] or domain [2] in the direction of decreasing h), we computed the value of g_0 (see [4]) for several parameter values and computed (for $n = 2$) the function

$$V(u) = T^n v - v \tag{2.16}$$

Our analysis showed that when the indicated passage occurs the fixed point of the transformation T becomes unstable and gives birth to the two stable fixed points of the transformation T^2 corresponding to stable periodic motion which is repeated after two pulses, and in which the impulses alternate with N ($N = 1,2$) oscillations of the oscillator.

Construction of function (2.16) for $n = 4$ showed that the bifurcation of a fixed point of the transformation T^2 (discussed above) for which the root of the characteristic polynomial passes through the value $\lambda = -1$ does exist, and that on passage through the corresponding bifurcation curve in the direction of decreasing h in the parameter space, the pair of fixed points of the transformation T^2 changes from stable to unstable. This is accompanied by the emergence from these points of the four stable fixed points of the transformation T^4 which correspond to stable periodic motion in which repetition occurs after four pulses, the pulses alternating with one ($N = 1$) oscillation of the oscillator.

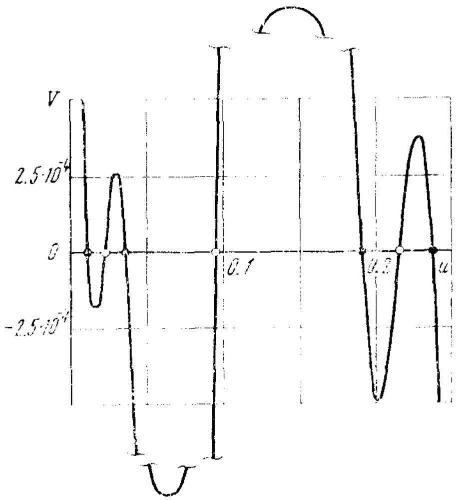

Fig. 7

Figure 7 shows the graph of the function $V(u)$ constructed for the parameter values $h^{1/4} = 0.5$, $r = 0.05865$. The zeros of the function indicated by the small circles in the Figure correspond to the unstable limit cycles (the middle dot corresponds to a single-turn limit cycle, the two extreme dots to a two-turn limit cycle); the zeros of the function indicated by dots in the figure correspond to a four-turn stable limit cycle.

Construction of function (2.16) for $n = 8$ showed that there exist bifurcations of a fixed point of the transformation T^4 for which the root of the characteristic polynomial passes through the value $\lambda = -1$, and that on passage through the corresponding bifurcation curve in the direction of decreasing h in the parameter space, the four fixed points of the transformation T^4 change from stable to unstable; this gives rise to the eight stable fixed points of the transformation T^8 corresponding to stable periodic motion in which repetition occurs after eight pulses, the pulses alternating with a single oscillation of the oscillator.

Let us cite the bifurcation values of the parameter r (for $h^{1/4} = 0.5$ and $N = 1$) which correspond to the birth of the stable fixed points of the transformation T^n ($n = 2,4,8$)

$$T^2 \text{ for } r = 0.05035, \quad T^4 \text{ for } r = 0.05836, \quad T^8 \text{ for } r = 0.05876$$

The above values of the parameter r define the ranges of existence of fixed stable points of the transformations T^2 and T^4, respectively,

$$0.05035 < r < 0.05836, \qquad 0.05836 < r < 0.05876$$

3. The viscous-friction model. 3.1. The point transformation. Let us consider the viscous-friction model (setting $r = 0$ in Eq. (1.5)). The oscillator receives a pulse in the segment s,

$$x = 0, \quad 0 < u_0 = Ke^{-2\pi H} < x^{\cdot} \leqslant K = u_1 \qquad H = H_1(1 - H_1^2)^{-1/2}$$

The transformation of the segment σ into the segment s ($v_1 \in \sigma$, $u \in s$) is of the following form:

$$v_1 = ue^{2\pi HN} \tag{3.1}$$

Curve (3.1) has N ($N = 1,2, \ldots$) branches (straight-line segments).

Let $D_N (u_0, v_1 (u_0, N))$ be the points of intersection of the various branches of curve (3.1) with the straight line $u = u_0 (v_1 (u_0,N) = Ke^{2\pi H(N-1)})$. The points $P_N (u_1, v_1 (u_0, N + 1))$ are then the points of intersection of the corresponding branches of curve (3.1) with the straight line $u = u_1$. As above, s_N are the half-intervals $v_1 (u_0, N) < v_1 < v_1 (u_0, N + 1)$ on the axis v_1.

Equation (3.1) defines the quantity u as a single-valued function of v_1 discontinuous at the points its $v_1 = v_1 (u_0, N)$; this definition is valid for all $v_1 > v (u_0, 1)$ and all u belonging to s .

Expressions (1.10) and (3.1) define the transformation T of the segment σ into itself. Let us consider the relative disposition of curve (1.10) and of the branches of surve (3.1) (straight-line segments) in the range $(u_0, u_1]$.

At the point of intersection of curves (1.10) and (3.1) we have

$$|dv/du| < |dv_1/du| \tag{3.2}$$

so that curve (1.10) can have a unique point of intersection with each branch of curve (3.1); this point of intersection corresponds to the stable fixed point of the transformation T.

Let W be the largest and w the smallest ordinate of curve (1.10) in the segment s. Inequality (3.2) is valid (see Appendix 6) for all points of curves (1.10) and (3.1) in the interval $[w, W]$, so that curve (1.10) can intersect not more than two distinct branches of curve (3.1) (the interval σ lies on not more than two half-intervals s_N and s_{N+1}) ;the transformation T^n $(n = 1, 2, \ldots)$ can have only stable fixed points (limit cycles of any multiplicity in the phase plane of the model under consideration can only be stable).

The following three variants of relative disposition of curve (1.10) and branches (3.1) are possible; curve (1.10) does not intersect any branch of curve (3.1); it intersects the $(N + 1)$-th branch; it intersects two branches, N and $N + 1$.

In the first case, making use of (3.2) and repeating the arguments of [5], we can show that the transformation T^n has n stable fixed points which correspond to stable periodic motion in which series of N and $N + 1$ oscillations of the oscillator alternate with the pulses.

In the second case there always exists a single stable fixed point of the transformation T which is associated in the phase plane with a spiral of $N + 1$ turns with closure by an instantaneous pulse; moreover, there can exist n stable fixed points of the transformation T^n (when the interval σ lies on the two half-intervals s_N and s_{N+1}). Depending on the initial conditions, a simple or complex stable periodic motion is established.

In the third case there always exist two simple stable fixed points of the transformation T. Depending on the initial conditions, we have either a periodic motion in which the pulses alternate with N oscillations, or a periodic motion in which $N + 1$ oscillations alternate with the pulses.

3.2. Structure of the parameter space. Now let us consider the parameter space H, h and K and isolate in it the domains corresponding to each of the three aforementioned variants of relative disposition of curve (1.10) and branches of curve (3.1).

For parameter values satisfying the condition

$$h = e^{-8\pi H} (e^{2\pi HN} - 1)(e^{2\pi HN} + 3)^3 K^4 \equiv G_1 (N, H) K^4 \tag{3.3}$$

curve (1.10) passes through the point D_N (cutting off equal segments of the straight line $u = u_0$ with the Nth branch of curve (3.1)).

For parameter values satisfying the condition

$$h = (e^{2\pi HN} - 1)(e^{2\pi HN} + 3)^3 K^4 \equiv G_2(N, H) K^4 \tag{3.4}$$

curve (1.10) passes through the point P_N (cutting off equal segments of the straight line $u = u_1$ with the Nth branch of curve (3.1)).

Equations (3.3) and (3.4) define two families of surfaces $\{\alpha_N\}$ and $\{\beta_N\}$ ($N = 1, 2, \ldots$), in the parameter space; these families of surfaces have the following properties (see Appendix 7): the surfaces within each of the families ($\{\alpha_N\}$ or $\{\beta_N\}$) do not intersect for $H > 0$ and $K > 0$; the surfaces of the respective families do intersect, and the surface $\{\alpha_{N+1}\}$ intersects only the surface $\{\beta_N\}$, but only once. The cross section of the parameter space H, h, K cut off by the plane $K = 1$ is shown in Fig. 8.

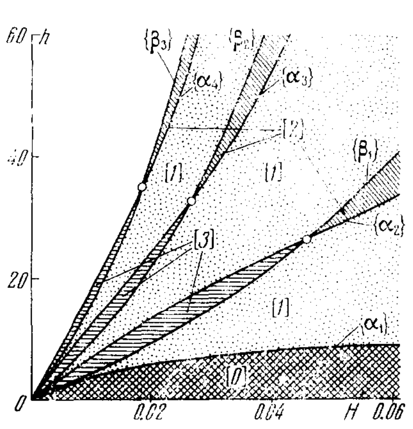

Fig. 8

The surface $\{\alpha_1\}$ bounds the domain [0] of parameter values for which curves (1.10) and (3.1) do not have points of intersection; here $\sigma \subset s$ and the clock does not work.

Each pair of surfaces $\{\alpha_{N+1}\}$ and $\{\beta_N\}$ ($N = 1, 2, \ldots$) defines domains [2] and [3] (Fig. 8 shows only three such pairs).

The domain [3] of parameter values lying below the surface $\{\alpha_{N+1}\}$ and above the surface $\{\beta_N\}$ corresponds to the first variant of relative disposition of curves (1.10) and (3.1) (they do not intersect). Curve (1.10) lies between the Nth and ($N+1$)-th branches of curve (3.1).

The domain [2] of parameter values situated above the surface $\{\alpha_{N+1}\}$ and below the surface $\{\beta_N\}$ corresponds to the third variant of relative disposition of curves (1.10) and (3.1). Curve (1.10) intersects the Nth and ($N+1$)-th branches of curve (3.1).

The whole remaining portion of the parameter space is occupied by the domains [1] which corresponds to the second variant of relative disposition of curves (1.10) and (3.1).

The domain [1] situated above domain [0] and below domains [3] and [2] defined by the surfaces $\{\alpha_2\}$ and $\{\beta_1\}$ corresponds to the case where the interval σ has points in common with the half-interval s_1 only (there exists only one stable fixed point of the transformation T, and this point corresponds to the point of intersection of curve (1.10) with the first branch of curve (3.1)).

For each fixed K the piece of the surface

$$h = 9K^4 e^{8\pi HN}, \qquad H \geqslant H_0 \equiv \frac{1}{2\pi(N+1)} \ln\frac{3}{2} \tag{3.5}$$

(here H_0 is the coordinate of the line of intersection of surface (3.5) with the surface $\{\alpha_{N+1}\}$) divides into two parts that part of domain [1] which lies above the domains [3] and [2] defined by the surfaces $\{\alpha_{N+1}\}$ and $\{\beta_N\}$ and below the domains [3] and [2] isolated by the surfaces $\{\alpha_{N+2}\}$ and $\{\beta_{N+1}\}$ ($N = 1, 2, \ldots$). The points of the domain which lie below the piece of the surface defined by (3.5) correspond to the case where the interval σ lies on the two half-intervals s_N and s_{N+1}, the points of the remaining part of the domain correspond to the case when the interval σ lies on the half-interval s_{N+1} only. In the second case there exists a unique stable fixed point of the transformation T which corresponds to the point of intersection of curve (1.10) with the ($N+1$)-th branch of curve (3.1); in the first case we can isolate (see Appendix 8) the domains $[1_2]$, $[1_3]$, ..., ..., $[1_n]$, each of which contains not only this fixed point, but also a stable fixed point of the transformations T^2, T^3, ..., T^n, respectively. The domain $[1_2]$ is bounded by the surface $\{\alpha_{N+1}\}$ and by the surfaces defined by the conditions

$$v_1(u_2) = v(u_1), \qquad v_1(u_3) = v(u_1)$$

(the surfaces $\{\gamma_1\}$ and $\{\gamma_2\}$) which are connected on surface (3.5) along the curve whose coordinate H can be determined from the equation

$$(2e^{2\pi H(N+2)} - 3)\,(2e^{2\pi H(N+2)} + 9)^3 = 9^3 e^{8\pi HN}$$

(the case where $u_2 = u_3 = u_{\min}$ and $v(u_{\min}) = v(u_1)$). The points of the surface $\{\gamma_1\}$ correspond to a phase space with a simple limit cycle whose zone of attraction consists of the entire phase plane with the exception of the countable set of points constituting the zone of attraction of the two-turn limit cycle.

Figure 9 shows a qualitative picture of the cross section of the parameter plane cut off by the plane $K =$ const in some neighborhood of the point of intersection of the surfaces $\{\alpha_2\}$ and $\{\beta_1\}$. The domains $[1_2]$, $[1_3]$, ..., $[1_n]$ are shown a "lobules" lying along $\{\alpha_2\}$; these lobules cumulate towards the point Γ.

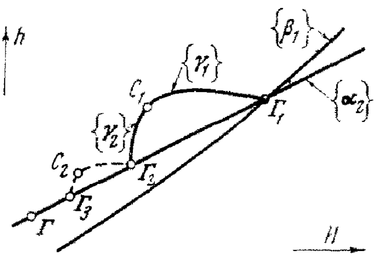

Fig. 9

The coordinates h of the surfaces γ and α for several values of H and $K = 1$ are as follows:

$H\,10^6 =$	33376	33565	33723	33957	34329
$\{\gamma_2\}$:	20.812	20.915	21.004	21.127	21.328
$\{\alpha_2\}$:	20.812	20.908	20.990	21.105	21.291
$H\,10^6 =$	34329	36935	39809	42837	44999
$\{\gamma_1\}$:	21.328	22.691	24.113	25.520	26.467
$\{\alpha_2\}$:	21.291	22.584	23.985	25.439	26.467

The coordinates $(H\,10^6, h)$ for the characteristic points of the domains $[1_2]$, $[1_3]$ and the point Γ are

$$\Gamma\,(32266,\ 20.250),\ \Gamma_3\,(32289,\ 20.262)\quad C_2\,(32311,\ 20.273)$$
$$\Gamma_2\,(33376,\ 20.812),\ C_1\,(34329,\ 21.328),\ \Gamma_1\,(44999,\ 26.467)$$

Appendix 1. From the first equation of (1.8) we obtain

$$b(\tau) = \frac{1}{\cos\tau}\{u\sin\tau + B^2[1 - \cos\tau - 2F(A,\tau) + F(2A,\tau)]\}$$

We note that $b(0) = 0$. For any $u =$ const we have

$$\frac{db}{d\tau} = \frac{u^2 + B^2\eta(\tau)}{\cos^2\tau} \equiv \frac{u^2 + B^2[\sin\tau - 2\Phi_1(A,\tau) + \Phi_1(2A,\tau)]}{\cos^2\tau}$$

$$\Phi_1(A,\tau) = \frac{1}{1+A^2}[A + e^{-A\tau}(\sin\tau - A\cos\tau)]$$

Since $\eta'(\tau) = (1 - e^{-A\tau})^2\cos\tau$ and $u > 0$, it follows that $db/d\tau > 0$, so that there exists a function $\tau = \tau(b)$ such that

$$\lim_{b\to 0}\tau(b) = 0$$

Appendix 2. For the functions $\varphi(\tau)$, $\xi(\tau)$ and $\psi(\tau)$ we have

$$\varphi(0) = \varphi'(0) = \varphi''(0) = \varphi'''(0) = 0, \quad \varphi^{IV}(0) = 2A^2$$
$$\xi(0) = \xi'(0) = \xi''(0) = \xi'''(0) = 0, \quad \xi^{IV}(0) = 6A^2$$

$$\psi(0) = \psi'(0) = \psi''(0) = 0, \quad \psi'''(0) = 2A^2 \qquad \text{(cont.)}$$

From the first expression of (1.8) we find that for $u = \text{const}$

$$\lim_{b\to 0} \frac{2b}{\sin\tau} = u + 1/4\, I_0 \qquad (A.1)$$

Hence, there exists the limit

$$\frac{3h}{A^2 8^3} = \lim_{b\to 0} b^3 B^2 = I_0 \lim_{b\to 0} \frac{\sin^3\tau}{\psi(\tau)} \lim_{b\to 0} \frac{b^3}{\sin^3\tau} = \frac{3\, I_0}{A^2} \lim_{b\to 0} \frac{b^3}{\sin^3\tau} \qquad (A.2)$$

Hence, (A.1) implies that

$$u = 1/4\, [(h/I_0)^{1/3} - I_0], \qquad v = \lim_{b\to 0} v(\tau) = 1/4\, [(h/I_0)^{1/3} + 3\, I_0] \qquad (A.3)$$

The condition $I_0 = \text{const}$ for which the limiting process ossurs is equivalent to the condition $h = \text{const}$. Hence, the limiting process for $b \to 0$, $B \to \infty$ and $h = \text{const}$ yields (A.3).

Appendix 3. At the point $P_1(2r\sqrt{6}, 4r\sqrt{5})$ we have $dv\, /\, du < 0$. Since

$$\frac{d}{dh}\left(\frac{dv}{du}\right) = -\frac{3u}{4v^3(v+3u)} < 0 \qquad (u = \text{const} > 0)$$

holds for curve (1.10), we recall (1.12) to conclude that $dv\, /\, du < 0$ also holds at the point P_N.

Appendix 4. The substitution $\zeta^2 = (N+1)(2N+3)$ transforms Eq. (2.14) into

$$2p^2 + p(1+8\zeta^2)^{1/2} + \zeta^2 - [(\zeta - \sqrt{3})(3\sqrt{3} + \zeta)^3]^{1/2} = 0$$

Here $\zeta \geqslant \sqrt{10}$, since $N \geqslant 1$. For p we obtain

$$4p = [1 + 8(\zeta - \sqrt{3})^{1/2}(3\sqrt{3} + \zeta)^{3/2}]^{1/2} - (1 + 8\zeta^2)^{1/2}$$

Let us consider the following function of the continuous argument ζ (q is a parameter):

$$f(\zeta, q) = (1 + 8\zeta^2 + 16q\zeta)^{1/2} - (1 + 8\zeta^2)^{1/2}$$

such that

$$f(0, q) = 0, \quad f_\zeta'(\zeta, q) > 0 \ (\zeta > 0), \quad \lim_{\zeta\to\infty} f(\zeta, q) = q\sqrt{8}$$

It is easy to show that for $\zeta \geqslant \sqrt{10}$ we have the inequalities

$$f(\zeta, {}^3/_2\sqrt{3}) < 4p < f(\zeta, 3\sqrt{2})$$

Since the relations

$$4 < f(\zeta, {}^3/_2\sqrt{3}), \qquad f(\zeta, 3\sqrt{2}) < 12$$

are valid for the function $f(\zeta, q)$, we have the following inequalities for p:

$$1 < p < 3$$

Appendix 5. As we know, the pair of fixed points of the transformation T^2 corresponds in the Lameray diagram to a rectangle whose vertices lie on the curves (1.10) and (2.1). Let us consider the case where curve (1.10) intersects only the branch $N = 1$ of curve (2.1). The rectangle with its vertex at the minimum of (1.10) ($\min v > \min v_1 = 2r\sqrt{6}$) corresponds to the pair of stable fixed points of the transformation T^2, and the rectangle with its vertex at the minimum of (2.1) ($\min v_1 > \min v = h^{1/4}/\sqrt{3}$) corresponds to the pair of unstable fixed points of the transformation T^2. Let us show that there exist parameter values for which both of the above possibilities are realized.

The condition under which there exists a rectangle whose vertex lies at the minimum of curve (1.10) is of the form

$$\{[16r^2 + {}^4/_{27}\sqrt{h} + 8r(r^2 + {}^4/_{27}\sqrt{h})^{1/2}]^{1/2} - [16r^2 + {}^1/_3\sqrt{h} - 8r(r^2 + {}^1/_3\sqrt{h})^{1/2}]^{1/2}\} \times$$
$$\times \{[16r^2 + {}^4/_{27}\sqrt{h} + 8r(r^2 + {}^4/_{27}\sqrt{h})^{1/2}]^{1/2} + 3[16r^2 + {}^1/_3\sqrt{h} - 8r(r^2 + {}^1/_3\sqrt{h})^{1/2}]^{1/2}\}^3 = h$$

This equation has a root in the range $r_1(z, h) / h^{1/4} < r / h^{1/4} < 1 / \sqrt{72}$ with respect to the variable $r / h^{1/4}$. The value $r / h^{1/4} = 1 / \sqrt{72}$ corresponds to a disposition of curves (1.10) and (2.1) such that $\min v = \min v_1$; the function $r = r_1(z, h)$ is such that $dv / dv_1 = -1$ ($r_1(z, h)$ is determined by expression (2.8).

The condition of existence of a rectangle whose vertex lies at the minimum of curve (2.1) is of the form

$$r\sqrt{24} - (16r^2 + \sqrt{h} - 8r\sqrt{r^2 + \sqrt{h}})^{1/2}]\,[r\sqrt{24} + 3(16r^2 + \sqrt{h} - 8r\sqrt{r^2 + \sqrt{h}})^{1/2}]^3 = h$$

This equation has a root in the range $1 / \sqrt{72} < r / h^{1/4} < 1/\sqrt{48}$ with respect to the variable $r / h^{1/4}$.

Appendix 6. The following cases are possible:

a) curve (1.10) is situated in such a way that $w > 2K$. We then have the inequalities

$$\max\left|\frac{dv}{du}\right| < 2 < \min\left|\frac{dv_1}{du}\right|$$

b) curve (1.10) is situated in such a way that $w < 2u$, $W < 2u$. Then

$$\max|dv/du| < 1 < \min|dv_1/du|$$

c) curve (1.10) is situated in such a way that $w < 2u < W$ (curve (1.10) intersects the straight line $v = 2u$) or $2u < w < 2K$, $W > 2K$. Since $|dv / du| \leqslant 2$, curve (1.10) in this case can intersect either the straight line $v = 2ue^{2\pi H}$ or the straight line $v = 2ue^{-2\pi H}$, i.e. we have either

$$\max|dv/du| < \max\{2 - {}^3/_2 e^{-4\pi H}, 1\} \quad (v < 2ue^{4\pi H})$$

or

$$\max|dv/du| < \max\{2 - {}^3/_2 e^{-2\pi H}, 1\} \quad (v < 2ue^{2\pi H})$$

We note that if $\max\{2 - {}^3/_2 e^{2\pi H}, 1\} = 1$, then the inequalities of Case (b) are fulfilled. Let us consider the case $H < {}^1/_2\,\pi\ln{}^3/_2$. In this case we have, respectively, either

$$\max|dv/du| < 2 - {}^3/_2 e^{-4\pi H} < 2 - e^{-2\pi H} < {}^1/_2 < \min|dv_1/du|$$

or

$$\max|dv/du| < 2 - {}^3/_2 e^{-2\pi H} < 1 < \min|dv_1/du|$$

In the case $H \geqslant {}^1/_2\pi\ln{}^3/_2$ only a branch of curve (3.1) for $N = 1$ can lie below the straight line $v = 2u$, and the inequalities of Case (a) or Case (b) are fulfilled.

Appendix 7. Let us consider the cross sections of surfaces (3.3) and (3.4) cut off by the planes $K = \text{const}$ (the curves in the plane Hh). Clearly,

$$G_1(N, H) < G_2(N, H), \quad G_1(N, H) < G_1(N+1, H), \quad G_2(N, H) < G_2(N+1, H)$$

Let us show that

$$G_2(N, H) < G_1(N+2, H) \tag{A.4}$$

Noting that

$$G_2(N, H) = e^{8\pi HN} + 8e^{6\pi HN} + 18e^{4\pi HN} - 27$$
$$G_1(N+2, H) = e^{8\pi H(N+1)} + 8e^{6\pi HN + 4\pi H} + 18e^{4\pi HN} - 27e^{-8\pi H} \tag{A.5}$$

we conclude that

$$G_1(N+2, H) - G_2(N, H) = e^{8\pi HN}(e^{8\pi H} - 1) + 8e^{6\pi HN}(e^{4\pi H} - 1) + 27(1 - e^{-8\pi H}) > 0$$

and inequality (A.4) is valid.

Now let us show that the equation

$$G_1(N+1, H) - G_2(N, H) = 0 \qquad (A.6)$$

has the unique positive root H^*. Making use of (A.5), we can reduce Eq. (A.6) to

$$f_2(H) - f_1(H) = 0$$

$$f_1(H) = 8e^{6\pi HN}(1 - e^{-2\pi H}) + 18e^{4\pi HN}(1 - e^{-4\pi H}), \quad f_2(H) = 27(1 - e^{-8\pi H})$$

for any fixed N ($N = 1, 2, \ldots$).

Since

$$f_1(0) = f_2(0) = 0,\ f_1'(0) = 88\pi,\ f_1'(H) > 0,\ f_1''(H) > 0$$

$$f_2'(0) = 216\pi,\ f_2'(H) > 0,\ f_2''(H) < 0, \quad \lim_{H\to\infty} f_1(H) = \infty, \quad \lim_{H\to\infty} f_2(H) = 27$$

it follows that Eq. (A.6) has a unique positive root H^*, and the following inequalities are valid:

$$G_2(N, H) < G_1(N+1, H)\ (0 < H < H^*), \quad G_2(N, H) > G_1(N+1, H)\ (H > H^*)$$

Now let us consider the form of the cross section of surfaces (3.3) and (3.4) for $K =$ const and fixed N. It is clear that in this case

$$dG_2/dH > 0,\ d^2G_2/dH^2 > 0, \quad dG_1/dH > 0, \quad \text{for} \quad N = 1, 2, \ldots$$

$$d^2G_2/dH^2 > 0 \quad \text{for} \quad N = 4, 5, \ldots$$

For $N = 1$ the cross section of surface (3.3) has a maximum and inflection point, and approaches the asymptote $h = K^4$ from above; for $N = 2, 3$ the cross sections have an inflection point.

Appendix 8. Let us consider passage in the parameter space through the piece of the surface $\{\alpha_{N+1}\}$ defined by the condition

$$H_0 \equiv \frac{1}{2\pi(N+1)} \ln \frac{3}{2} < H < H^*$$

for increasing h.

Here H^* is the coordinate of the line of intersection of the surfaces $\{\alpha_{N+1}\}$ and $\{\beta_N\}$. For the points of the piece in question we have

$$v_1(u_0, N+1) = ke^{2\pi HN} = v(u_0) < v(u_1), \quad \min v(u) < v_1(u_0, N+1)$$

For any arbitrarily small increase of the parameter h curve (1.10) has a point of intersection with the $(N+1)$-th branch of the curve (3.1) and intersects the straight line $v = v_1(u_0, N+1)$ $(u_2 < u_{\min} < u_3)$ at two points $R_1(u_2, v_1(u_0, N+1))$ and $R_2(u_3, v_1(u_0, N+1))$.

By virtue of (1.12), for each fixed value of the parameter K there exists a value of H on the piece in question such that $v_1(u_3) = v(u_1)$.

The equation of the surface which yields the corresponding bifurcation curve on intersecting surface (3.5) is

$$h = (Ke^{2\pi HN} - u)(Ke^{2\pi HN} + 3u)^3$$

where u is the root of the equation

$$(Ke^{2\pi HN} - u)(Ke^{2\pi HN} + 3u)^3 - (ue^{2\pi H(N+1)} - K)(ue^{2\pi H(N+1)} + 3K)^3 = 0$$

This curve and $H = H^*$ isolate a domain of the indicated piece of the surface $\{\alpha_{N+1}\}$ where $v_1(u_3) > v(u_1)$. Hence, crossing the surface $\{\alpha_{N+1}\}$ through this domain, we enter the domain $[1_2]$ for which $v_1(u_2) < v(u_1)$, $v_1(u_3) > v(u_1)$.

Considering the relative disposition of curves (1.10) and (3.1) which corresponds to the domain $[1_2]$, we find that the transformation T^2 transforms the segment $v_1(u_2) \leqslant v < v(u_1)$ into part of it. Hence, the compressing transformation T^2 has a stable fixed point.

Carrying out a similar analysis, we can show that below the piece of surface (3.5) in the domain [1] there is a domain $[1_n]$ which corresponds to stable fixed points of the transformations T and T^n.

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REGIONS OF STABILITY IN A CASE CLOSE TO THE CRITICAL ONE

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V. G. VERETENNIKOV
(Moscow)

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A system of real differential equations

$$x_s^{\cdot} = \mu_s x_s - \lambda_s y_s + X_s(x, y), \qquad y_s^{\cdot} = \mu_s y_s + \lambda_s x_s + Y_s(x, y)$$
$$x \equiv (x_1, \ldots, x_n), \quad y \equiv (y_1, \ldots, y_n) \quad (s = 1, \ldots, n) \tag{0.1}$$

is considered. Here μ_s are small, positive real parts of the complex-conjugate roots of the characteristic equation. X_s and Y_s are holomorphic functions of x_s and y_s, and their expansions begin with the terms of at least second order.

The definition of the stability of motion in the cases close to the critical ones given in [1] and the results of [2] extended to cover the case of n pairs of pure imaginary roots are used to establish the regions of stability for the system (0.1).

1. To be able to apply the Kamenkov [3, 4] transformation to our study of the stability of the system (0.1) for all $\mu_s = 0$ in the nonresonant cases, we consider the equivalent problem on the stability of the system

$$\rho^{\cdot} = 2\rho^{k+1} R_0(z) + 2\rho^{k+2} R_1(z) + \ldots$$
$$z_s^{\cdot} = 2\rho^k z_s (z_1 R_{s1}^{(2)} + z_2 R_{s2}^{(2)} + \ldots + z_n R_{sn}^{(2)}) + \ldots$$
$$R_{sj}^{(2)} = R_s^{(2)} - R_j^{(2)}, \quad R_0 = \sum z_j R_j^{(2)}(z) \quad (z_1 + \ldots + z_n = 1,\ s = 1, \ldots, n) \tag{1.1}$$

We can use the Kamenkov [3] theorem on instability as the basis for asserting that the unperturbed motion is unstable if at least one, nontrivial, real solution of the system of